

**FIELD THEORY AND
THE PHENOMENON OF TURBULENCE**

V. Gurarie

*Department of Physics, Princeton University
Princeton, NJ 08544, USA***ABSTRACT**

We study the phenomenon of turbulence from the point of view of statistical physics. We discuss what makes the turbulent states different from the thermodynamic equilibrium and give the turbulent analog of the partition function. Then, using the soluble theory of turbulence of waves as an example, we construct the turbulent action and show how one can compute the turbulent correlation functions perturbatively thus developing the turbulent Feynman diagrams. And at last, we discuss which part of what we learnt from the turbulence of waves can be used in other types of turbulence, in particular, the hydrodynamic turbulence of fluids. This paper is based on the talk delivered at SMQFT (1993) conference at the University of Southern California.

1. Introduction

The phenomenon of turbulence has been known to physicists for more than a century and yet it remains to be one of the unsolved problems of modern physics. Its formulation is extremely simple. The behavior of incompressible fluid is governed by two simple equations¹

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \nabla) \mathbf{v} \right) = -\mathbf{grad} \, p + \nu \Delta \mathbf{v} \quad (1.1)$$

$$\text{div } \mathbf{v} = 0$$

for the unknowns \mathbf{v} and p , the velocity of the fluid at a given point and the pressure. For a given problem one just needs to solve these equations with the appropriate boundary conditions to find the motion of the fluid completely, as it seems. But as was known for a long time, these so called laminar solutions describe what we observe only when the velocity is small enough. As the velocity increases, the solutions for the Eq. (1.1) become unstable and the fluid switches to a new regime of a very complex motion with the velocity pulsating almost randomly and without any noticeable order. To describe what exactly is going on when the fluid is in such regimes is of extreme importance to both the fundamental physics and the applications.

Since its discovery, the enormous amount of efforts has been put into the studying of turbulence^{1,2}. It was realized that the most simple situation where the turbulence exhibits itself is when we somehow shake the fluid homogeneously in its entire volume. This is called the homogeneous turbulence^{1,2} and this is what we are going to study here.

Of course, the first natural thing to do when approaching this problem is to realize that the random pulsations of the velocity leave us with little else to do other than to pass to the statistical description. Instead of the velocity \mathbf{v} we should work with the average $\langle \mathbf{v} \rangle$. The average here is supposed to be over the time. We can also define the correlation functions like $\langle \mathbf{v}(\mathbf{x})\mathbf{v}(\mathbf{y}) \rangle$. This is exactly what is done in statistical physics when studying complex systems with many degrees of freedom. But statistical physics is a well developed branch of science. It teaches us that the averages over the time should be replaced by the averages over the space with the probability distribution. The probability distribution is usually found on the one of the first pages of any book on statistical physics. It is simply

$$P = \exp\left(-\frac{E}{T}\right) \quad (1.2)$$

where E is the energy of the system, and T is called the temperature. Eq. (1.2) is called Gibbs distribution. And so as long as the energy is known, we are left just with a mathematical problem of computing the averages of P over the phase space. For the motion described by Eq. (1.1) with the absence of the viscosity $\nu = 0$ the energy can be shown to be¹

$$E = \int \frac{\mathbf{v}^2}{2} \quad (1.3)$$

and the integral over the phase space turns out to be a Gaussian path integral. In fact, the phase space unit volume should also be properly defined before doing the actual calculations, but all those difficulties can be overcome and the correlation functions can be computed exactly (see, for example, the reference [4]).

However, it does not mean we solved the problem of turbulence. The turbulent motion of fluid is not described by Gibbs distribution. It is because the motion of fluid is essentially dissipative. The dissipation is described by the term $\nu \Delta \mathbf{v}$ and is very important for the whole phenomenon. While the energy conserving case $\nu = 0$ is properly described by the thermodynamic equilibrium distribution given above, switching on dissipation, even infinitesimally weakly, completely changes the whole picture. Since in real life all the motion is with dissipation, the thermodynamic calculations are of little importance to physics of fluids, apart from a few very special cases like the atmosphere of Jupiter etc.⁴

What really makes the picture is that the constant motion of fluid must be supported by some external force which injects the energy into the fluid. This energy is dissipated with the same rate as it is injected to provide for some form of a dynamic equilibrium.

In this form the problem again looks almost unapproachable. It looks as if the correlation functions essentially depend on how exactly we inject the energy and

what kind of stirring force we use to do the job. But already in twenties it was realized that there was still some universality left in the phenomenon^{1,2}.

The energy consumption depends on $\Delta \mathbf{v}$ and that means, the smaller the wavelength, the bigger the dissipation. It may turn out (and it actually does) that the energy dissipation is unimportant at wavelengths bigger than a certain small value, while the stirring force acts only at some large wavelengths characteristic for a given source of energy. In between there is what is called the inertial range where we may neglect both the stirring force and the viscosity. The situation here is reminiscent of that of the phase transition theory. There is some cutoff (two cutoffs in our case) and a universal picture in the middle. In the inertial range the Eq. (1.1) with $\nu = 0$ works, but yet we cannot use the thermodynamic distribution because we observe a flux of energy from large to small scales through the inertial range. All the energy comes to a given wave from a larger one, and it goes away to smaller ones unlike the thermodynamic equilibrium where the detailed balance principle is satisfied, that is there is as much energy transferred from one wave to another as there is energy transferred back. This picture of energy cascade through the system is characteristic of turbulence. In fact, when we will be speaking of turbulence in this paper, we will mean the systems with fluxes of energy or other conserved quantities. This is what really defines the phenomenon of turbulence and makes it so difficult to study.

There exist some statistical states which are characterized by the fluxes which are just small deviations from the thermodynamic equilibrium. These are studied in the kinetics. However, they have nothing to do with turbulence. The turbulent states are far away from the thermodynamic equilibrium and the notion of temperature has no sense for them⁶.

Kolmogorov was the first to attempt a quantitative study of this picture. He said that the only relevant parameter which could enter any correlation function was the energy transfer rate per unit mass ϵ (if we express everything in terms of the unit mass, then the density of fluid ρ should not enter any formula). Then some of the correlation functions can immediately be found by a simple dimensional analysis. For example, let us find the energy distribution over different wavelengths $E(k)$. Its dimension is $\frac{cm^2}{s^2}cm$ because $E(k)dk$ must have the dimension of the energy per unit mass. ϵ is the energy per unit mass per unit time, so its dimension is $\frac{cm^2}{s^2}\frac{1}{s}$. By comparing the dimensions, we immediately obtain

$$E(k) = \frac{1}{2} \langle \mathbf{v}(k) \mathbf{v}(-k) \rangle \propto \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \quad (1.4)$$

This is called the Kolmogorov spectrum. Its coordinate version can be written as

$$\langle (\mathbf{v}(\mathbf{x}) - \mathbf{v}(0))^2 \rangle \propto x^{\frac{2}{3}} \quad (1.5)$$

which implies the dimension of the velocity to be $-\frac{1}{3}$. The Kolmogorov spectrum was measured in many experiments, numerical and “real-life” and was found to be very close to $-\frac{5}{3}$ although there is no particular agreement on the exact value of that number. The proximity of the measured value of the spectrum to the predicted one has persuaded many researchers that $-\frac{5}{3}$ is in fact an *exact* value and many efforts

had been spent in the attempts to prove it. And yet, even now, 50 years after Eq. (1.4) was first written, we do not know if it is exact or not. The reason for us to doubt the Eq. (1.4) is that the considerations which lead us to it are essentially mean field theory. We now know very well from the study of critical phenomena that mean field theory often fails⁵ because the cutoffs may enter explicitly the formulae we are trying to find. That is why we are still in need of constructing a quantitative theory of turbulence not based on any conjectures.

Most of the attempts of proving or disproving Eq. (1.4) were based on the following idea, which is sometimes referred to as Wyld's approach. We introduce a random external force \mathbf{f} in the Eq. (1.1),

$$\begin{aligned} \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \nabla) \mathbf{v} \right) &= -\mathbf{grad} \, p + \nu \Delta \mathbf{v} + \mathbf{f} \\ \text{div } \mathbf{v} &= 0 \end{aligned} \tag{1.6}$$

and we presume that the random force is Gaussian with the correlation function

$$\langle \mathbf{f}(\mathbf{x}, t) \mathbf{f}(\mathbf{y}, t') \rangle = \mathbf{D}(\mathbf{x} - \mathbf{y}) \delta(t - t') \tag{1.7}$$

where \mathbf{D} is some function[†]. Then we solve Eq. (1.6) perturbatively in powers of nonlinear interaction term and apply the renormalization group technique to interpret this solution as the renormalization of viscosity and the coefficient in front of the interaction term (which is 1 in Eq. (1.6)). The hope is then to pass somehow to the limit when the function \mathbf{D} acts only at large distances thus reproducing the behavior of the fluid in the turbulence conditions. However, all the calculations performed in this fashion showed that in this limit the equations of motion tend to the nontrivial fixed point of renormalization group which cannot be treated perturbatively or even worse, everything diverges to meaningless infinities⁷. So some other technique should be employed when studying turbulence.

In this paper, following mainly the reference [8], we shall try to revive the old approach to turbulence which assumes the existence of the stationary probability distribution. We shall study the wave turbulence, a theory which, unlike hydrodynamics, can be solved perturbatively, and we shall find the turbulent probability distribution for it. We shall see that the turbulent probability distribution is essentially different from the thermodynamics one, and the hope is, some of the features we shall examine are independent of the particular model choice and are present in any kind of turbulence.

2. Probability Distributions

2.1 The Turbulent Distributions

In this section we attempt to explain why Gibbs distribution cannot describe the

[†] Note that here the average is understood in the sense different from that of Eq. (1.4) because it is not an equal-time correlation function, but rather a time dependent one. Also, here we induce a randomness on the system with the random external force, while in turbulence the randomness should be generated by the equation itself. Later throughout the paper we are going to study only time independent equal time correlation functions which are generated by a very complicated motion of the fluids.

turbulent states of fluids or other statistical systems. Let us review the derivation of Gibbs distribution. It usually starts with claiming that any stationary probability distribution should be of the form $\exp(-F)$ where F is the additive integral of motions, as follows from the Liouville theorem³. Then they claim the only additive integral of motion the system can have is the energy E , that is how Eq. (1.2) is obtained. In our case Eq. (1.2) does not work as we already learnt, so something must have gone wrong with the above reasoning. What really had gone wrong was there were other integrals of motion⁹. In fact, any dynamical system has as many integrals of motion as the number of its degrees of freedom, at least locally in time. To be more explicit, the initial conditions expressed in terms of the changing variables give us a formal integral of motion.

To be more precise, let us imagine we have a hamiltonian system with the Hamiltonian $H(P, Q)$, P and Q being the momenta and the coordinates. The equations of motion are

$$\begin{aligned}\frac{dP}{dt} &= -\frac{\partial H}{\partial Q} \\ \frac{dQ}{dt} &= \frac{\partial H}{\partial P}\end{aligned}\tag{2.1.1}$$

One can show that the Eq. (1.1) with the viscosity $\nu = 0$ are also hamiltonian⁶, so the motion of the incompressible inviscid fluid is a particular case of the Eq. (2.1.1).

Let us now imagine we somehow solved the Eq. (2.1.1) and found the coordinates and the momenta and the functions of time and the initial values of the coordinates and the momenta, or

$$\begin{aligned}P &= P(Q_0, P_0; t) \\ Q &= Q(Q_0, P_0; t)\end{aligned}\tag{2.1.2}$$

with

$$\begin{aligned}P(Q_0, P_0; 0) &= P_0 \\ Q(Q_0, P_0; 0) &= Q_0\end{aligned}\tag{2.1.3}$$

Then we can invert the above functions to get

$$\begin{aligned}Q_0 &= Q_0(Q, P; t) \\ P_0 &= P_0(Q, P; t)\end{aligned}\tag{2.1.4}$$

The functions Q_0 and P_0 we thus obtained are the integrals of motion *by definition*. They explicitly depend on time, but we can go on to solve for t the first of the equations (2.1.4) and substitute that to the second one. Or alternatively, we can try to pass to the limit $t \rightarrow \infty$.

We can suspect a strange paradox here since the integrals of motion (2.1.4) exist for any systems, even for nonintegrable ones which, by definition, should not possess too many integrals of motion! The resolution to this paradox lies in the fact that, after the elimination of time, those integrals of motion will become nonsinglevalued functions, depending on the expressions like $\arctan(P/Q)$ which are called the angle variables in mechanics. Yet they can perfectly be used to construct the probability distributions other than thermodynamic equilibrium for the systems with infinite numbers of degrees of freedom⁸.

What is most exciting about those integrals of motion is that some of them give us the turbulent distributions, that is those distributions describe the systems with fluxes of energy or other conserved quantities, while the rest are anomalously not conserved. To search for the right probability distribution from so many ones given by Eq. (2.1.4), we must solve the kinetic equation, which can be understood as an anomaly cancellation condition. We shall discuss it in the next section of this paper.

Of course, to construct those integrals for Eq. (1.1) is a hopeless task. To do that, we would need to solve those equations for any initial data, while it is clearly impossible to do. Here we want to try to achieve a more modest aim, namely to find those integrals for the turbulence of waves.

We end our discussion of turbulence on general terms at this point. In what follows, we are going to discuss only the turbulence of waves unless otherwise stated.

2.2 Wave Turbulence

The turbulence of waves is the flux-states of the systems consisting of waves with a small interaction⁶. Its Hamiltonian can be written down in the form

$$H = \sum_p \omega_p a_p^\dagger a_p + \sum_{p_1 p_2 p_3 p_4} \lambda_{p_1 p_2 p_3 p_4} a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \quad (2.2.1)$$

It is just a collection of waves with the energy spectrum ω_p and the four wave interaction $\lambda_{p_1 p_2 p_3 p_4}$ with the evident properties $\lambda_{p_1 p_2 p_3 p_4} = \lambda_{p_2 p_1 p_3 p_4} = \lambda_{p_3 p_4 p_1 p_2}$. H is a classical Hamiltonian and a_p are classical variables while a_p^\dagger are just their complex conjugates. Let us note that this Hamiltonian conserves the total wave number

$$N = \sum_p a_p^\dagger a_p. \quad (2.2.2)$$

We could also have considered the three wave interaction⁶, but we are not going to do it here.

The alluring property of this system is that the interaction is small, which allows us to use the perturbation theory in all the calculations. On the other hand, there are many real systems existing in Nature which are described by Eq. (2.2.1), the most notable of them being the gravitational waves on the surface of water.

This system can be in the state of the thermodynamic equilibrium, which would be described by the Gibbs distribution

$$P = \exp\left(-\frac{H + \mu N}{T}\right). \quad (2.2.3)$$

But it could also exhibit the turbulent behavior. Let us imagine there is a source of energy at some small value $p = a$ and a sink at $p = \Lambda \gg a$. Then there will be a flux of energy in between these two values of p . (Actually there have to be two sinks, to absorb both the energy and the wave number, but it is irrelevant to the present discussion.)

To find the probability distributions which are more general than Eq. (2.2.3), we need to find the integrals of motion more general than H and N . And to do that,

we need to solve the equations of motion. It can be done perturbatively in powers of the interaction. However, there is a simpler way to achieve this aim.

Let us look for the integrals of motion in the form of the infinite series

$$F = \sum_p f_p a_p^\dagger a_p + \sum \Lambda_{p_1 p_2 p_3 p_4} a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} + \quad (2.2.4)$$

$$+ \sum \Omega_{p_1 p_2 p_3 p_4 p_5 p_6} a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger a_{p_4} a_{p_5} a_{p_6} + \dots$$

Here Λ , Ω , etc. are some still unknown functions. We impose the condition on F that it is an integral of motion, or $\{HF\} = 0$, $\{$ and $\}$ being the Poisson brackets. With the help of the obvious definition of the Poisson brackets,

$$\{a_p, a_{p'}^\dagger\} = i\delta_{pp'}$$

we find those functions to be

$$\Lambda_{p_1 p_2 p_3 p_4} = \frac{f_{p_1} + f_{p_2} - f_{p_3} - f_{p_4}}{\omega_{p_1} + \omega_{p_2} - \omega_{p_3} - \omega_{p_4}} \lambda_{p_1 p_2 p_3 p_4} \quad (2.2.5)$$

$$\Omega_{p_1 p_2 p_3 p_4 p_5 p_6} = 4 \sum_{p_7} \frac{(\lambda_{p_7 p_1 p_5 p_6} \Lambda_{p_2 p_3 p_4 p_7} - \Lambda_{p_7 p_1 p_5 p_6} \lambda_{p_2 p_3 p_4 p_7})}{\omega_{p_1} + \omega_{p_2} + \omega_{p_3} - \omega_{p_4} - \omega_{p_5} - \omega_{p_6}}, \quad (2.2.6)$$

and so on. We can in principle find recursively all the terms in the series (2.2.4), one after another.

However, in this form F is not very well defined. The denominators of $\Lambda_{p_1 p_2 p_3 p_4}$, $\Omega_{p_1 p_2 p_3 p_4 p_5 p_6}$ and, as one can check, of all the higher order terms in the series (2.2.4) may be equal to zero for certain values of the indices p_1 , p_2 , etc. Then the sums in (2.2.4) are not defined.

If we had the finite number of degrees of freedom, that would mean the destruction of the integrals F . This phenomenon is well known in mathematical physics¹⁰, the destruction of the integrals meaning that the system is no longer integrable after the interaction was turned on.

However, here we assume that we have an infinite number of a_p and a_p^\dagger . Then the infinite number of degrees of freedom means the sums in (2.2.4) turn into the integrals. We can shift the poles in (2.2.5) and (2.2.6) to the complex plane to define

$$\Lambda_{p_1 p_2 p_3 p_4} = \frac{f_{p_1} + f_{p_2} - f_{p_3} - f_{p_4}}{\omega_{p_1} + \omega_{p_2} - \omega_{p_3} - \omega_{p_4} - i\epsilon} \lambda_{p_1 p_2 p_3 p_4} \quad (2.2.7)$$

$$\Omega_{p_1 p_2 p_3 p_4 p_5 p_6} = 4 \sum_{p_7} \frac{(\lambda_{p_7 p_1 p_5 p_6} \Lambda_{p_2 p_3 p_4 p_7} - \Lambda_{p_7 p_1 p_5 p_6} \lambda_{p_2 p_3 p_4 p_7})}{\omega_{p_1} + \omega_{p_2} + \omega_{p_3} - \omega_{p_4} - \omega_{p_5} - \omega_{p_6} - 2i\epsilon}. \quad (2.2.8)$$

where ϵ is a small constant. Then the integrals the sums of Eq. (2.2.4) turned into become well defined. We should also shift the poles in all the higher terms of Eq. (2.2.4). Of course, we must pass to the limit $\epsilon \rightarrow 0$ in all the calculations.

Let us note that we wrote $2i\epsilon$ in the denominator of Eq. (2.2.8). To understand why we should do so, we must resort to the original way of finding F as explained in the previous subsection. We start by assigning the variables a_p and a_p^\dagger the initial

values $a_p(t=0) = a_p^0$, $a_p^\dagger(t=0) = a_p^{\dagger 0}$. Then by solving the equations of motion one can find $a(t)$ and $a^\dagger(t)$ as functions of a^0 and $a^{\dagger 0}$ and time. By inverting those functions, one can find a^0 and $a^{\dagger 0}$ as functions of time, $a(t)$ and $a^\dagger(t)$. The quantity

$$F = \sum_p f_p a_p^{\dagger 0} a_p^0, \quad (2.2.9)$$

where f_p are some arbitrary coefficients, is an additive integral of motion by definition. Now if we try to solve the equations of motion following from (2.2.1) perturbatively in powers of interaction, we shall find that as soon as we express F in terms of a_p and a_p^\dagger it turns into something close to Eq. (2.2.4). The only difference will be the explicit presence of time in form of the coefficients $\frac{1 - \exp(-i\omega t)}{\omega}$. To obtain the time independent integrals of motion, we shall pass to the limit $t \rightarrow \infty$ using the well known formula

$$\lim_{t \rightarrow \infty} \frac{1 - \exp(-i\omega t)}{\omega} = \lim_{\epsilon \rightarrow 0} \frac{1}{\omega - i\epsilon} \quad (2.2.10)$$

As soon as we do that to Eq. (2.2.9) it will turn itself into (2.2.4). This treatment shows that ϵ is actually an inverse time cutoff.

The pole shift to the complex plane in construction of the integrals of motion is very important for the proper description of turbulence. If one computes F in the first two orders of perturbation theory by solving the equations of motion and passing to the limit $t \rightarrow \infty$, one notices that the denominator of (2.2.6) is formed by adding up the denominators of (2.2.5) and that is why we put $2i\epsilon$ in there. By induction, it is possible to prove that we will get $ni\epsilon$ in the denominator of the n th term of Eq. (2.2.4). Writing $ni\epsilon$ is very essential for consistency of some of the calculations⁸, even though we shall not encounter such calculations in this paper.

2.3. Turbulent Partition Function

The integrals of motion F are the most general integrals the system described by Eq. (2.2.1) can have. By choosing different functions f_p we can obtain different integrals of motion including the ones we knew before.

For example, if $f_p = 1$, then Eq. (2.2.4) turns into

$$F = \sum_p a_p^\dagger a_p \quad (2.3.1)$$

which is just the wave number quantity N . If we choose $f_p = \omega_p$, then F will simply coincide with the Hamiltonian. The corresponding probability distribution

$$\exp(-F) \quad (2.3.2)$$

will just be the Gibbs one.

However, if we choose different f_p we shall obtain completely new distributions. We shall see later that some of them describe the turbulent states with constant fluxes while the rest will describe nonstationary statistical states.

The turbulent partition function is given by

$$Z = \int \prod_p da_p \prod_p da_p^\dagger \exp(-F) \quad (2.3.3)$$

and any correlation function can be computed using

$$\langle X \rangle \equiv \frac{1}{Z} \int \prod_p da_p \prod_p da_p^\dagger X \exp(-F) \quad (2.3.4)$$

The turbulent partition function looks not too much different from the standard statistical one. Nevertheless, it possesses many unusual properties. First, the turbulent F is an infinite series in powers of the interaction and each next term is not independent of the previous ones, but rather has a special form. In fact, only the first term of the series (2.2.4) is arbitrary, all other terms are defined by the first one. Secondly, we need to employ the pole shifting trick to define F properly and we have to pass to the limit $\epsilon \rightarrow 0$ in the final answers. Nevertheless, one can check by taking the complex conjugate of F that it is explicitly real so we shall not obtain unphysical complex values for real correlation functions. Yet, the pole shifting will play a very important role in the whole technique.

We would like now to turn to the discussion of what makes some of F describe the constant fluxes.

3. Kinetic Equation

3.1. Kinetic Equation in the first order of perturbation theory

Gibbs distribution describes the thermodynamical states which do not depend on time. One can check that explicitly with

$$\left\langle \frac{dX}{dt} \right\rangle = \langle \{HX\} \rangle = \int \{HX\} \exp(-H) = \int \{HH\} X \exp(-H) \equiv 0 \quad (3.1.1)$$

where we took the integral by parts and used that $\{HH\} \equiv 0$. So, the average of derivative with respect to the time is always zero. A really amazing fact is that the probability distribution F can give rise to the correlation functions which do depend on time.

Let us for example compute the average of the wave number change with respect to the time, or

$$\left\langle \frac{d(a_p^\dagger a_p)}{dt} \right\rangle \quad (3.1.2)$$

The equation expressing this quantity in terms of the functions f_p and equating that with zero is called the kinetic equation.

By computing the Poisson bracket of the wave number with the Hamiltonian we obtain

$$\begin{aligned} \left\langle \frac{d(a_p^\dagger a_p)}{dt} \right\rangle &= \langle \{H, a_p^\dagger a_p\} \rangle = \\ &= -i \sum_{p_1 p_2 p_3 p_4} \lambda_{p_1 p_2 p_3 p_4} \langle a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \rangle (\delta_{pp_1} + \delta_{pp_2} - \delta_{pp_3} - \delta_{pp_4}) = 0 \end{aligned} \quad (3.1.3)$$

Since the function F is real, one can easily see from (3.1.3) that only the imaginary part of the correlation function $\langle a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \rangle$ will contribute to the kinetic

equation which can actually be written down in the form

$$-4i \sum_{p_2 p_3 p_4} \lambda_{pp_2 p_3 p_4} \text{Im} \langle a_p^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \rangle = 0 \quad (3.1.4)$$

Let us find the four point correlation function perturbatively. Its zero order value is equal to

$$\begin{aligned} \langle a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \rangle &\equiv \frac{1}{Z} \int \prod_p da_p \prod_p da_p^\dagger a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \exp(-\sum_p f_p a_p^\dagger a_p) = \\ &= \frac{1}{f_{p_1} f_{p_2}} (\delta_{p_1 p_3} \delta_{p_2 p_4} + \delta_{p_1 p_4} \delta_{p_2 p_3}) \end{aligned}$$

This expression is real so it will not contribute to Eq. (3.1.4). Now the first order contribution to that function is obtained by expanding the exponent in powers of λ and is given by

$$-4\lambda_{p_1 p_2 p_3 p_4} \frac{f_{p_3} + f_{p_4} - f_{p_1} - f_{p_2}}{\omega_{p_3} + \omega_{p_4} - \omega_{p_1} - \omega_{p_2} - i\epsilon} \frac{1}{f_{p_1} f_{p_2} f_{p_3} f_{p_4}} \quad (3.1.5)$$

Imaginary part of Eq. (3.1.5) is

$$4i\pi\lambda_{p_1 p_2 p_3 p_4} (f_{p_1} + f_{p_2} - f_{p_3} - f_{p_4}) \delta(\omega_{p_1} + \omega_{p_2} - \omega_{p_3} - \omega_{p_4}) \frac{1}{f_{p_1} f_{p_2} f_{p_3} f_{p_4}}$$

and after substituting it to (3.1.4) we arrive at the kinetic equation in the first order of perturbation theory

$$16\pi \sum_{p_2 p_3 p_4} \lambda_{pp_2 p_3 p_4}^2 \delta(\omega_p - \omega_{p_2} - \omega_{p_3} - \omega_{p_4}) \frac{f_p + f_{p_2} - f_{p_3} - f_{p_4}}{f_p f_{p_2} f_{p_3} f_{p_4}} = 0.$$

or, taking into account the infinite number of degrees of freedom,

$$16\pi \int dp_2 dp_3 dp_4 \lambda_{pp_2 p_3 p_4}^2 \delta(\omega_p - \omega_{p_2} - \omega_{p_3} - \omega_{p_4}) \frac{f_p + f_{p_2} - f_{p_3} - f_{p_4}}{f_p f_{p_2} f_{p_3} f_{p_4}} = 0. \quad (3.1.6)$$

The Eq. (3.1.4) or its first order approximation Eq. (3.1.6) is of special importance. The probability distributions F do not always give the stationary states of the systems we study. Only when we choose f_p such that Eq. (3.1.4) is satisfied, the corresponding F is a true turbulent distribution. So, to solve the turbulence of waves means two things. One is to solve the kinetic equation and find such f_p that it is satisfied. The second is to find the correlation functions using F with those f_p we found.

Let us note that the thermodynamic equilibrium provides the simplest solution because the four point correlation function used in Eq. (3.1.4) is always real when computed with $F \equiv H$ or $F \equiv N$. Really, the only place where complexity can come into play is the term $i\epsilon$ and it appears neither in H nor in N . In the language of the first order equation Eq. (3.1.6) $f_p = 1$ or $f_p = \omega_p$ makes the integrand of Eq. (3.1.6) to be zero. The integrand has an evident physical meaning of the wave exchange between fixed wave numbers.

However, there are other solutions of Eq. (3.1.4) which give us nonzero fluxes. They can be found explicitly in the first order of perturbation theory by solving the Eq. (3.1.6)⁶. They occur when the integrand of Eq.(3.1.6) is nonzero while the integral is zero, providing the equilibrium without the detailed balance. As soon as we solve the Eq. (3.1.4) and find uniquely the function f_p and therefore the function F as well, we can start computing correlation functions and the problem of turbulence will be at least formally solved.

Now as is well known in the phase transition theory, the first order approximation does not always give the right answer. The first order approximation is called mean field theory in the physics of phase transitions and is called *weak turbulence* in our theory. However, there are important cases where we must sum up all orders of perturbation theory⁸. They are called *strong turbulence*. Weak turbulence has been well studied in the last two decades. Strong turbulence is much more difficult to study. It is especially alluring to study the strong case because the original problem of turbulent fluid does not have any small parameter and as such is the problem of strong turbulence.

We shall postpone further discussion of weak and strong turbulence till next section.

3.2. The Anomalous Nature of Kinetic Equations

We saw in the previous subsection that the distributions F can in general describe nonstationary statistical states. That can easily lead to the apparent paradox. We can repeat the line (3.1.1) for the distribution F .

$$\langle \frac{dX}{dt} \rangle = \langle \{HX\} \rangle = \int \{HX\} \exp(-F) = \int \{HF\} X \exp(-F) \equiv 0$$

because $\{HF\} = 0$. But that clearly contradicts the derivation of the kinetic equation. We saw that the average

$$\langle \frac{d(a_p^\dagger a_p)}{dt} \rangle$$

is not equal to zero apart from the specially chosen f_p .

The resolution of this paradox lies in the fact that F is the integral of motion only in the limit $\epsilon \rightarrow 0$. If we compute $\{HF\}$ we shall discover that it is of the order of ϵ . But then, when computing the integral $\int \{HF\} X \exp(-F)$, we shall find that it in itself is of the order of $1/\epsilon$. Those epsilons cancel and we can safely pass to the limit $\epsilon \rightarrow 0$ to get the finite answer for the kinetic equation. The effect is completely analogous to the anomaly of Quantum Field Theory. By imposing the cutoff ϵ on the theory we violate the conservation of F and it is never fully restored.

The kinetic equation can thus be interpreted as a condition for anomaly cancellation. As soon as it is fulfilled, F becomes a true integral of motion, even within this regularization scheme.

The anomaly shows how important those ϵ 's are for the theory. While the distribution F we found works only for wave turbulence, that may be a general

feature of all the turbulent distributions including those for Eq. (1.1).

4. Turbulent Field Theory

4.1. Diagrammatic technique

To be able to go beyond the first order of perturbation theory, we must learn how to compute correlation functions in arbitrary order. The function F contains arbitrary high powers of λ and it is not so easy to expand $\exp(-F)$. However that can be done in general. It is possible to formulate the result of the expansion in terms of relatively simple Feynman rules, though more complex than the standard ones⁸.

F contains interactions of the fourth, fifth and all the higher orders, but it turns out it is enough to use the four-vertices in the diagrams because the higher order terms like (2.2.6) are effectively expressed in terms of the four point one.

Since the quadratic term in F has the form $-f_p a_p^\dagger a_p$, the propagator has the form $1/f_p$. The interaction coefficient is $\lambda_{p_1 p_2 p_3 p_4}$, it will “live” at each vertex of the diagram. The most nontrivial part is a prefactor coming from a very complex form of the interaction. The rules of computing that prefactor are not too simple and are given in reference [8]. Here we shall give that expression just for one diagram to give a general idea of what should be expected.

Let us consider, as an example, the simplest bubble diagram one can think of in the φ^4 -like theory. Let us call the momenta of its external legs as 1, 2, 3, and 4, and of its two intermediate lines as 5 and 6. Its expression will be

$$\begin{aligned} & \frac{\lambda_{1234}\lambda_{1256}\lambda_{5634}}{f_1 f_2 f_3 f_4 f_5 f_6} \times \\ & \times \left(\frac{(f_1 + f_2)(f_5 + f_6)}{(\omega_1 + \omega_2 - \omega_3 - \omega_4 - 2i\epsilon)(\omega_5 + \omega_6 - \omega_3 - \omega_4 - i\epsilon)} + \right. \\ & + \frac{(f_3 + f_4)(f_5 + f_6)}{(\omega_1 + \omega_2 - \omega_3 - \omega_4 - 2i\epsilon)(\omega_1 + \omega_2 - \omega_5 - \omega_6 - i\epsilon)} - \\ & \left. - \frac{(f_1 + f_2)(f_3 + f_4)}{(\omega_1 + \omega_2 - \omega_5 - \omega_6 - i\epsilon)(\omega_5 + \omega_6 - \omega_3 - \omega_4 - i\epsilon)} \right) \end{aligned} \quad (4.1.1)$$

The expression begins with the standard product of propagators and vertices. But then we see a prefactor which is completely different from what was encountered in Field Theory before. It was the higher order vertices generated by F which combined in the nontrivial way to give rise to that prefactor. While it looks a little horrible, it does have at least one nice feature though, it goes to 1 if $f_p \rightarrow \omega_p$ or if the turbulent distribution goes into the thermodynamic one. The “turbulent” field theory we obtained can thus be considered a direct generalization of the standard Field Theory techniques.

Let us review the differences of the turbulent field theory from the standard one. Its action is a nonlocal expression given by an infinite series given by Eq. (2.2.4). There is an additional ϵ -regularization so in effect the turbulent action is a generalized function. The correlation functions may depend on time and the cancellation

of that dependence gives us additional condition, called the kinetic equation, to be satisfied for the action to describe the turbulence. And the diagrams obtained from the perturbative expansion differ from standard ones by a special prefactor.

This theory has not been studied well yet. One may ask what “classical” equations of motion its action corresponds to, if many notions of field theory like the existence of Hilbert space of states and others can be applied to it. And of course whether it can be constructed in cases what the perturbation theory does not work. These are the questions to be answered in future.

4.2. ϵ -expansion

Now it is time to discuss how to actually compute something meaningful in this theory. The program is clear. For a given ω_p and $\lambda_{p_1 p_2 p_3 p_4}$ we must solve the equation (2.1.4) for f_p and then, for the f_p found we must compute the correlation functions which may be of interest. The most interesting of them is the wave spectrum

$$n_p = \langle a_p^\dagger a_p \rangle \quad (4.2.1)$$

To find the solutions of the equation (3.1.4) or even of its first order approximation (3.1.6) which are different from the thermodynamic equilibrium is a very difficult task. The only approachable case is when the spectrum ω_p and the interaction λ are scale invariant

$$\omega_p = p^\alpha, \quad \lambda_{p_1 p_2 p_3 p_4} = (p_1 p_2 p_3 p_4)^{\frac{\beta}{4}} U \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \quad (4.2.2)$$

where U depends only on the ratio of the momenta and their mutual angles. Then we can apply the Zakharov’s transformations⁶ to solve (3.1.6) to get

$$f_p = p^\gamma, \quad \gamma = \frac{2}{3}\beta + d \text{ or } \gamma = \frac{2}{3}\beta + d - \frac{\alpha}{3} \quad (4.2.3)$$

where d is the number of space dimensions. The first of the solutions corresponds to the energy flux while the second one to the flux of particle number.

While the exact derivation of Eq. (4.2.3) is rather complicated, there is a nice way of interpreting it. Let us take the Eq. (3.1.6) and make a change of variables in the integral

$$p_2 = x_2 p, \quad p_3 = x_3 p, \quad p_4 = x_4 p \quad (4.2.4)$$

We shall get

$$\frac{\partial n_p}{\partial t} = p^{-3\gamma+2\beta+3d-1-\alpha} 16\pi \int dx_2 dx_3 dx_4 K(x_2, x_3, x_4) \delta(1 + x_2^\alpha - x_3^\alpha - x_4^\alpha) \frac{1 + x_2^\gamma - x_3^\gamma - x_4^\gamma}{(x_2 x_3 x_4)^\gamma} = 0. \quad (4.2.5)$$

where K is

$$K(x_2, x_3, x_4) = p^{-2\beta+d} \int d\phi_2 d\phi_3 d\phi_4 \lambda_{pp_2 p_3 p_4}^2,$$

$d\phi_2$, $d\phi_3$, and $d\phi_4$ are the angle integrations associated with $d^d p_2$, $d^d p_3$, and $d^d p_4$ respectively. It may not be evident at a first glance, but by construction, K does not depend on p .

Let us compare Eq. (4.2.5) with the continuity equation

$$\frac{\partial n_p}{\partial t} + \frac{\partial J_p}{\partial p} = 0 \quad (4.2.6)$$

We see that the wave flux J_p is

$$J_p = -p^{-3\gamma+2\beta+3d-\alpha} \frac{X}{-3\gamma+2\beta+3d-\alpha} \quad (4.2.7)$$

We denoted the integral in Eq. (4.2.5) by X . This integral is just a number which is not dependent on p . The condition $J_p = \text{const}$ (or $\omega_p J_p = \text{const}$ for the energy spectrum $\omega_p n_p$) coincides with (4.2.3). The denominator of Eq. (4.2.7) is then equal to 0, but so is X .

However, (4.2.3) must be regarded as just the first approximation to the answer, just as mean field theory is just the first approximation for the critical phenomenon correlation functions and they often give the wrong critical indices⁵. The criterion for the applicability of the mean field theory techniques is, as always, the dimension of the interaction constant λ_0 . We can easily find it by analyzing F with $f_p = p^{-\gamma}$ as a zero approximation. We obtain

$$\kappa = -\dim(\lambda_0) = \beta + d - \gamma - \alpha \quad (4.2.8)$$

or, after substituting Eq. (4.2.3),

$$\kappa = \frac{\beta}{3} - \alpha \quad \text{or} \quad \kappa = \frac{\beta}{3} - \frac{2}{3}\alpha \quad (4.2.9)$$

The most striking difference between this parameter and the ϵ -parameter of the phase transition theory is that it does not depend on the number of space dimensions at all. Also, the convergence or divergence of the integrals we encounter in the computations is governed not only by the power counting but also by the parameter U of the interaction in Eq. (4.2.2). We may always choose U so as to make all integrals to be always convergent, for example, by taking U to be exponentially decaying function at large ratio of momenta.

However, recently there was an attempt by the author⁸ to study the ϵ -expansion for the case when $U=1$. Then the interaction does not influence the convergence or divergence of the integrals at all. If $\kappa > 0$ then mean field theory is exact (weak turbulence), while if $\kappa < 0$ (strong turbulence), then it had been shown that taking into account higher diagrams computed according to the rules mentioned above we can compute a correction to the mean-field spectrum γ in the form

$$\gamma \rightarrow \gamma - \frac{2}{3}\kappa \quad (4.2.10)$$

when κ is a small negative number and $\alpha < 1$.

5. Open problems

We have studied in this paper the probability distribution for the turbulent states of the statistical systems. While we did that for the turbulence of waves,

we hope that some of the distinctive features of the distribution we discovered are universal.

Let us return now to the discussion of the main task of solving the original problem of the hydrodynamics turbulence.

We saw that the Hamiltonian formulation was rather important for the discussion of turbulence. There exists a Hamiltonian formulation of hydrodynamics, the so-called Clebsch variables. We define

$$\mathbf{v} = \mathbf{grad} \varphi + \lambda \mathbf{grad} \mu \quad (5.1)$$

The field φ is a dependent variable due to the incompressibility equation $\text{div } \mathbf{v} = 0$. Then it turns out the variables λ and μ are the canonical ones, with the hamiltonian given by Eq. (1.3) where we substitute Eq. (5.1) for the velocity. It can be rewritten in the momentum space as

$$H = \int_{p_1 p_2 p_3 p_4} \lambda_{p_1 p_2 p_3 p_4} a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \quad (5.2)$$

where a and a^\dagger are the Fourier transforms of $\lambda + i\mu$ and $\lambda - i\mu$ and $\lambda_{p_1 p_2 p_3 p_4}$ are a specifically chosen the interaction coefficient (see reference [6] for the derivation of the fluid motion reformulation in terms of the canonical variables)

We see that the Hamiltonian (5.2) looks like the wave Hamiltonian given by Eq. (2.2.1) without the wave part, $\omega_p = 0$. One can ask if we can somehow interpret the turbulent field theory we discussed above in the limit $\omega \rightarrow 0$. The straightforward limit is impossible because each higher order diagram will contain a higher power of ω in its denominator. Another method is to introduce “mean field” frequencies¹¹ by

$$\omega_p \propto \int_q dq \lambda_{pqpq} \langle a_q a_q^\dagger \rangle.$$

Not surprisingly, this immediately leads to the Kolmogorov spectrum as an answer, as any mean field theory should do.

The only hope of honestly taking that limit, we believe, lies in conformal field theory. In two dimensions the turbulent field theory gives a set of the scale invariant correlation functions and it is possible they are conformally invariant as well. That may be directly checked, however, in case of the turbulence of waves, which would be very interesting to do and which has not been done yet.

Another important task here is to understand better the behavior of the infrared modes. The turbulent correlation functions grow with distance. Let us take for example the function Eq. (1.5). Of course, its validity is something to be proved, but yet the future full theory will probably give just a small correction to the index $-\frac{5}{3}$ of Eq. (1.4).

The Eq. (1.5) shows there is a strong infrared divergence of any integrals involving the velocity correlation functions, and that means everything depends on what is going on at the boundary. Those divergencies prevented a complete construction of conformal theory of two dimensional fluid turbulence⁹.

A similar divergency occurs in the wave turbulence theory discussed in this paper. There $\gamma > d$ and because of that there is an infrared divergency there as well. One can actually see that this IR divergency appears in many of the integrals in the perturbative calculations. There has not been, to the author's knowledge, any attempts to interpret them in one way or another.

A nice way of thinking of those divergencies is by assuming the infrared modes of the theory form some kind of a condensate, like the Bose condensate of the weakly interacting Bose gas. The overall infrared motion of the waves acts as a background for the small fluctuations around it. This picture can be checked quantitatively in the theory of wave turbulence. The idea would be to fix the amplitude of the zero mode $a_0 = A$ like in Bose gas condensation theory, substitute it into Eq. (2.2.4), and see whether there is a choice of A which cancels the IR divergencies. It is still unknown if this or a similar computation can provide an explanation of the behavior of zero modes. However we do know from numerical experiments that large scale motion is very important for the hydrodynamic turbulence. In two dimensions we observe a few large vertices of fluid motion and the turbulence is a fluctuation around those vertices. In three dimensions we observe a condensate of vertex filaments. A future theory of turbulence must be able to account for those effects and it could be its formulation depends crucially on taking them into account correctly.

The last thing we would like to mention is the absence of any nonperturbative exactly solvable models in turbulence which is very frustrating. All theories solved here exactly thus far had been trivial. On the other hand, in field theory and statistical mechanics a large number of nontrivial exactly solvable models has been discovered which by far increased our understanding of the physics of field theory and their study is one of the most important activities now in this field. One may hope that soon the theory of turbulence will also benefit from the discovery of such models.

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